Transistion from Laminar to Turbulent Flow in Pipes

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In pipeline design, for which one needs a means of ascertaining whether the flow will be laminar or turbulent, the Reynolds number is the criterion for Newtonian fluids. The principal purpose of this study was to formulate a more general criterion to characterize the flow regime and to test this form in application to non-Newtonian fluids.

Intuitive physical arguments suggested the use of a local stability parameter which is a function of the ratio of input energy to energy dissipation for an element of fluid. If the parameter is applied to a Newtonian fluid in laminar pipe flow, one finds that it has a maximum value of 0.385 times the critical Reynolds number, or 808. As the criterion is presumed to be general, it is inferred that the value of 808 defines the boundary between stable laminar and stable turbulent pipe flow for all fluids. The inference has been varified for several pseudoplastic fluids.

Osborne Reynolds (*1*) and those who followed his lead established that the dimensionless group subsequently named after Reynolds

$$N_{\mathrm{Re}} = (DV\rho/\mu)$$

is for Newtonian fluids flowing isothermally in straight, smooth, circular ducts, a criterion of type of flow, laminar or turbulent. When the Reynolds number is less than a critical value of about 2,100, laminar flow exists; transient flow disturbances are damped out. When the value of the Reynolds number is greater than critical, turbulent flow is usually encountered. Although under special conditions laminar flow can be produced at supercritical Reynolds numbers, it is metastable. A transient disturbance can induce turbulent flow, which will not revert to laminar flow unless the Reynolds number is reduced below 2,100.

The Reynolds number applies only to Newtonian fluids whose rheological character is described by the coefficient of viscosity. As distinct regimes of laminar and turbulent flow are also observed for other kinds of fluids, it is to be inferred that the Reynolds number is a special form of a more general criterion.

DEVELOPMENT OF CRITERION

The search for the flow criterion starts along the classical approach to the theoretical study of laminar flow stability, in which small perturbation theory is applied to the Navier-Stokes and continuity equations (for example reference 2). Concerned only with Newtonian fluids, this approach as usually followed has not produced the flow criterion sought. It does however, as developed below, suggest the forms of perturbation energy supply and dissipation terms.

The equations of motion (precursors of the Navier-Stokes equations, surface stresses not expressed in terms of shear rates) can be written as

$$\left.\begin{aligned} \rho\frac{Du}{Dt} &= \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} \\ \rho\frac{Dv}{Dt} &= \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} \end{aligned}\right\} \quad (1)$$

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The equations are written in two dimensions for a fluid on which no body forces act. The x and u direction is parallel to the axis. As small regions are to be considered, Cartesian coordinates are used in place of the less convenient cylindrical coordinates. The first subscript on the stress symbol indicates the face of a cubical element on which the stress acts; the second subscript indicates the direction of action. It is evident that σ_{xx} and σ_{yy} are normal stresses; σ_{xy} and σ_{yx} which are equal are shear stresses.

When no disturbance is present, steady state values are to be used:

$$u = U, \qquad v = 0, \qquad \sigma_{ij} = \tau_{ij}$$

When a small disturbance is present,

$$u = U + u'$$
$$v = v'$$
$$\sigma_{ij} = \tau_{ij} + \tau_{ij}'$$

The energy equation is produced by taking the scalar product of the vector velocity and the equations of motion in vector form. The energy equation for the steady state is subtracted from that for the disturbed state leaving significant first-order perturbation terms:

$$\begin{aligned} \rho\frac{D}{Dt}(Uu') - \left\{U\frac{\partial\tau_{xx}'}{\partial x} + u'\frac{\partial\tau_{xx}}{\partial x}\right\} &= -\rho v' U\frac{\partial U}{\partial y} \\ &\quad + \left\{U\frac{\partial\tau_{yx}'}{\partial y} + u'\frac{\partial\tau_{yx}}{\partial y}\right\} \end{aligned} \quad (2)$$

The terms on the left side of the equation represent the time rate of increase, unit-volume basis, of surplus energy in the disturbed region. The first term on the right represents the rate at which energy is supplied; the remaining terms represent the rate at which it is dissipated.

To carry this analysis further one would need to relate shear stress to shear rate, that is to specify the rheological nature of the fluid and thus lose generality. Also one would need to define the form of the perturbation stream function, which is not known for the kinds of disturbances of interest. Carried this far, however, the analysis does suggest that the energy-dissipation term takes the form of a perturbation in the quantity

$$-u\frac{\partial\sigma_{yx}}{\partial y} \quad (3)$$

The energy-supply term can be arrived at in another way if the radial transport of axial momentum across a unit surface normal to the y direction is considered. There results a tangential force on the unit surface $-\rho v'u$ which, when multiplied by the velocity gradient, gives the rate of perturbation energy supply from the base flow. In first-order perturbation form the term is

$$-\rho v' U\frac{\partial U}{\partial y} \quad (4)$$

The fluid transported radially moves from a position where the dissipation rate is given by Equation (3) to a position where the rate is greater by the amount

$$\frac{\partial}{\partial y}\left(-u\frac{\partial\sigma_{yx}}{\partial y}\right)\delta y$$

In first-order perturbation form the term is

$$\frac{\partial}{\partial y}\left(-U\frac{\partial\tau_{yx}}{\partial y}\right)\delta y \quad (5)$$

The ratio of Equations (4) and (5) is proposed as the stability index. Elimination of the remaining perturbation factors δy and v' is necessary. It is noted that their ratio is a time interval, and it is postulated that for effective disturbances this time must be related to a characteristic time of the undisturbed flow, which is taken as the reciprocal of the velocity gradient. The substitution of $\partial U/\partial y$ is therefore made for $-v'/\delta y$. Finally it is noted that

$$\frac{\partial\tau_{yx}}{\partial y} = -\frac{\tau_w}{r_w}$$

The stability parameter is, with these changes

$$Z = \frac{r_w\rho U}{\tau_w}\frac{\partial U}{\partial y} \quad (6)$$

Inspection shows that Z is zero at both the pipe wall and the center line and has a maximum value at an intermediate position. The maximum value can be obtained if the steady state velocity is known as a function of radial position. For a Newtonian fluid

$$Z_{\max} = \sqrt{\frac{4}{27}}\frac{DV\rho}{\mu}, \quad \text{at} \quad \frac{r}{r_w} = \frac{1}{\sqrt{3}}$$

Thus the maximized stability function meets the necessary condition that the Reynolds number be a special form of it.

As the critical value for the Reynolds number is 2,100, it is assumed that the critical value of Z_{max} above which turbulent flow is to be expected is, for all nonthixotropic, true fluids

$$Z_c = 2{,}100\sqrt{\frac{4}{27}} = 808$$

A stability parameter similar to Z was proposed by Rouse (*10*), who, from dimensional analysis and the recognition that the parameter should have zero value at both the axis and the pipe wall, suggested the stability parameter

$$\frac{y^2\rho}{\mu}\frac{\partial U}{\partial y}$$

This parameter, meaningful only for Newtonian fluids, maximizes at r/r_w of 1/3, the maximum value being 8/27 of the Reynolds number.

Clearly the maximized form of Rouse's parameter is equivalent to the Reynolds number. Its validity as a general parameter has been tested in two ways. In the first the coefficient of viscosity was interpreted, for non-Newtonian fluids, as the ratio of shear stress to rate of shear, and the parameter was tested in the same manner as Z in the next section. It did not successfully predict the onset of turbulence in pseudoplastic fluids.

A second test consists in the prediction of where, in the cross section of the stream, laminar flow of a Newtonian fluid is least stable. Rouse predicts the critical r/r_w as 0.333; the authors predict it as 0.577. Gibson (*11*), who observed dye filaments in water, reported that the critical ratio is 0.58 for large pipes, increasing somewhat as pipe diameter decreases below about 2 in. From his data one would estimate a ratio of 0.64 for standard ½-in. pipe. A similar conclusion is inferred from the work of Leite and Kuethe (*12*), who produced air flow at supercritical Reynolds numbers, introduced a disturbance, and made hot-wire anemometer measurements downstream from the source of the disturbance. At a short distance downstream, where the parabolic profile was still well approximated, they observed maximum fluctuations in the anemometer signal at r/r_w of 0.6.

It should be mentioned that Leite and Kuethe and others have observed that a stream at supercritical Reynolds number sometimes will not become turbulent when a finite disturbance is introduced. Their observations appear to be at variance with an essential premise of the derivation outlined above, namely that a very small disturbance can grow and produce turbulence. There is no contradiction if only certain kinds of small disturbances are effective; then one might speculate further that effective kinds are probably always present in the complex vibrations to which industrial piping is subjected.

TEST OF CRITERION

An attempt has been made to verify the criterion for pseudoplastic fluids. It is assumed that the behavior of these fluids is described by the power law

$$\tau_{yx} = K\left(\frac{\partial U}{\partial y}\right)^n \qquad (7)$$

The expression for Z_{max} becomes

$$Z_{max} = \frac{r_w^2\rho\Gamma^2}{\tau_w}\,\phi(n),$$

at

$$\frac{r}{r_w} = \left(\frac{1}{n+2}\right)^{n/n+1}$$

where

$$\phi(n) = \frac{(3n+1)^2}{n}\left(\frac{1}{n+2}\right)^{n+2/n+1} \qquad (8)$$

Here use is made of the flow function, which proves to be a convenient variable in the analysis of many problems of rheology. If the critical value of 808 is used

$$\tau_{wc} = \frac{r_w^2\rho\Gamma_c^2}{808}\,\phi(n) \qquad (9)$$

Equation (9) can be solved together with the analogue of the Hagen-Poiseuille Law

$$\tau_w = K\left(\frac{3n+1}{n}\right)^n\Gamma^n \qquad (10)$$

which applies to laminar flow, to yield the maximum flow rate (as Γ) at which laminar flow is stable.

If this method of verification is to be employed, Γ, τ_w pipe-flow data are needed.

Pipe-flow data were obtained in a horizontal pipe system differing only slightly from that described by Christiansen, Ryan, and Stevens (*3*). Clean steel pipes of ½-, ¾-, and 1-in. nomial diameter were employed. The pressure drop was measured over a 10-ft. length following an 8-ft. calming section. Flow rate and pressure drop were measured, and Γ was computed from the former and τ_w from the latter as

$$\tau_w = \frac{r_w}{2}\left(-\frac{\Delta p}{\Delta x}\right)$$

Some flow data were also taken in ¾-in. Pyrex pipe. Viscometer data were obtained in the rotating viscometer described by Stevens (*4*), a rotating-shell instrument with a shell radius of 3.808 cm., a clearance of 0.155 cm., and a shearing surface height of 16.2 cm.

The fluids used in the experimental work were carboxy-methyl cellulose (CMC) in water. Additional pipe-flow data were obtained from the literature: on 52% rock solids in water from Wilhelm, Wroughton, and Loeffel (*5*); on 23% yellow clay in water from Caldwell and Babbitt (*6*); on Carbopol in water and Attasol clay in water from Dodge (*7*).

Plots on logarithmic paper of experimental values of τ_w and Γ are shown on Figures 1 through 6. From the portion of these plots in the laminar-flow range n was determined as the slope and $\phi(n)$ computed for use in Equation (10) with experimental values of r_w to determine the τ_{wc} vs. Γ_c lines (dashed lines) on Figures 1 through 6. Each slope value (n) was measured from the best straight line through all data for laminar flow. This measurement is not critical because, as shown in the table of f_c values, $\phi(n)$ is a very weak function of n.

The test of the criterion is in how precisely the transition to turbulence, as

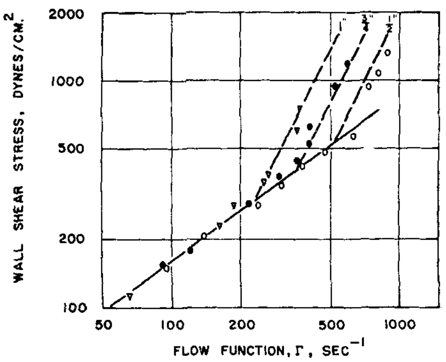

Fig. 1. Wall stress vs. flow rate for 1.0% carboxy methyl cellulose in water (*9*).

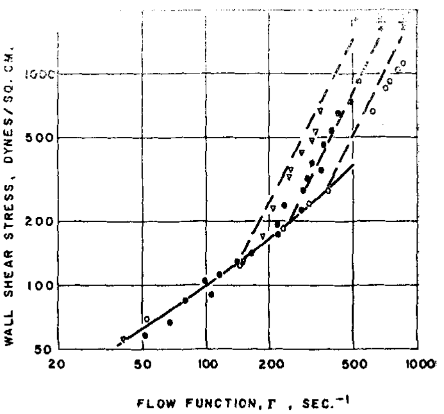

Fig. 2. Wall stress vs. flow rate for 0.75 and 0.78% carboxy methyl cellulose in water (*9*).

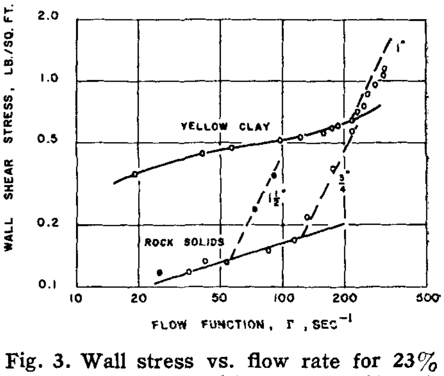

Fig. 3. Wall stress vs. flow rate for 23% yellow clay in water (*6*) and for 52% rock solids in water (*5*).

indicated by the experimental data, is predicted by the intersection of the appropriate calculated $\tau_{wc} - \Gamma_c$ (dashed) lines with the laminar flow line [equivalent to solving Equations (9) and (10) simultaneously]. It is seen that the predicted transition conditions agree reasonably well with the observed conditions. This test of the criterion is not however

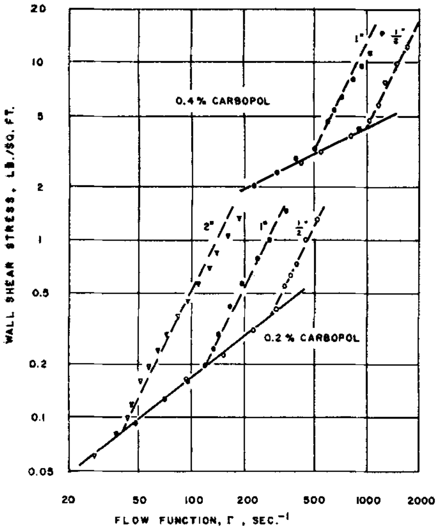

Fig. 4. Wall stress vs. flow rate for Carbopol in water (7).

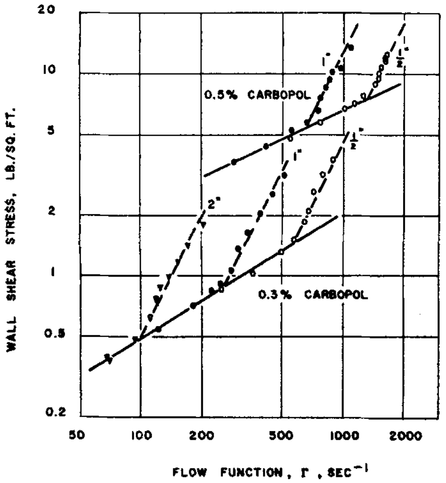

Fig. 5. Wall stress vs. flow rate for Carbopol in water (7).

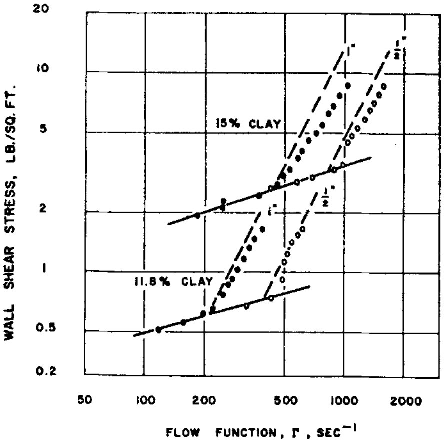

Fig. 6. Wall stress vs. flow rate for Attasol clay in water (7).

conclusive, partly because the pipe-flow data lack the desired precision and partly because the assumption that the fluids conform to the power law, Equation (7), is only approximate. The dashed lines are not intended to represent the data for turbulent flow; for most pseudoplastic fluids they do not do so over a large range of flow rates.

An interesting observation can be made relative to the suggestion of Metzner and Reed (8) that the value of the friction factor may be the same for all fluids at the critical condition. By definition the friction factor is given by the expression

$$f = \frac{2\tau_w}{r_w^2 \rho \Gamma^2} \qquad (11)$$

For fluids obeying the power law, Equation (7), one may combine Equations (8) and (11), setting $Z_{max} = 808$ to obtain

$$f_c = \frac{\phi(n)}{404}$$

Values of f_c calculated from this equation are

n	$f_c = \frac{\phi(n)}{404}$
1.0	0.0076
0.8	0.0072
0.6	0.0068
0.4	0.0066
0.2	0.0075

It is seen that $\phi(n)$ is rather insensitive for the useful range $0.2 < n < 1$. If the criterion proposed here is correct, the suggestion of Metzner and Reed is an approximation suitable for most engineering work.

SUMMARY

From largely intuitive arguments, guided by the theory of laminar-flow stability, a criterion of flow type in straight tubes of circular cross section has been developed:

$$Z = \frac{r_w \rho U}{\tau_w}\frac{\partial U}{\partial y}, \quad \text{at} \quad \frac{dZ}{dy} = 0$$

Application requires knowledge of the laminar velocity profile, which can be derived from the rheological law for the fluid of interest. The value of this criterion at laminar-turbulent transition is 808, laminar flow being predicted for values less than, and turbulent flow for values greater than, 808.

The criterion has been verified analytical for Newtonian fluids and experimentally by means of pipe-flow data for several pseudoplastic liquids, the reohological behavior of these liquids being approximated by the power law.

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NOTATION

Force (F), length (L), time (T) dimensions given

- D = tube diameter, L
- f = friction factor, dimensionless
- K = contant in power law, Equation (7)
- n = constant in power law, Equation (7)
- N_{Re} = Reynolds number, dimensionless
- p = pressure, FL^{-2}
- Q = volumetic flow rate, L^3T^{-1}
- r = radial distance from tube axis, L
- t = time
- u = local axial velocity, LT^{-1}
- U = local axial velocity, undisturbed flow, LT^{-1}
- v = local radial velocity, positive when toward axis, LT^{-1}
- V = mean velocity, LT^{-1}
- x = distance in axial direction, L
- y = distance from wall, normal to axis, L
- Z = stability parameter, dimensionless

Greek Letters

- Γ = flow function $Q/\pi r_w^3$, T^{-1}
- ρ = density, FT^2L^{-4}
- σ = stress, FL^{-2}
- τ = stress, FL^{-2}
- τ_w = sheaf stress
- μ = coefficient of viscosity, FTL^{-2}

Subscripts

- c = value at critical condition
- w = value at tube wall
- x, y = direction indexes
- $'$ = perturbation quantities

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